MOTION AND HEAT TRANSFER IN TUBES OF LIQUIDS WITH TEMPERATURE-DEPENDENT VISCOSITIES

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The problem of the velocity and temperature distribution of a liquid flowing in a cylindrical tube in the case where the viscosity varies exponentially with temperature is considered.

1. There have been many theoretical and experimental studies of hydrodynamics and heat transfer in a laminar flow of viscous liquids in tubes [1]. A problem of great practical interest is that of the motion of a viscous liquid in nonisothermal conditions, where the solution entails great difficulties of an analytical nature.

A flow with variable viscosity was investigated in [2] for a problem similar to the Graetz–Nusselt problem. In [2] the inertial and convective terms in the equations of motion and energy were only partially taken into account by averaging over the thickness of the thermal boundary layer. In this case, however, the analytical integration of the equations could be effected only by methods of numerical analysis, and the obtained results were valid only in the region of small reduced lengths. A solution suitable for the whole heat transfer region was obtained by the Karman–Pohlhausen integral method in [3]. In [4] heat transfer and resistance in a flow of gas with temperature-dependent properties were calculated by a finite-difference method and some interpolation formulas, predicting the results of calculation to within 3%, were proposed. In this case the system of differential equations of motion, continuity, and energy were investigated in the boundary-layer approximation.

The problem of heat transfer and hydraulic resistance of a viscous incompressible liquid in the region of stabilized heat transfer in the case of boundary conditions of the second kind was investigated in [5].

We can also mention the interesting investigations of some thermodynamic problems [6, 7] which led to the discovery of the hydrodynamic thermal explosion. Similar results indicating the existence of a critical combination of parameters at which a steady flow is impossible can evidently be expected in the theory of convective heat transfer with variable physical properties.

In the present paper we investigate the heat transfer and hydraulic resistance of a viscous incompressible liquid in a round tube in the case of boundary conditions of the second kind. We will assume that hydrodynamic and thermal processes are steady, and that the viscosity varies with temperature in accordance with the interpolation equation

$$\mu = \mu_0 \exp[-\beta (T - T_0)] \tag{1.1}$$

Here μ_0 is the viscosity of the liquid on entry into the heat-transfer section, and β is a parameter which depends on the kind of liquid and the temperature range.

If the temperature gradients are small, formula (1.1) gives a statisfactory agreement with experimental data. The equations of motion and heat transfer in the case of constant thermal diffusivity and straight-line particle trajectories have the form [8]

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial r}\left(\mu \frac{dv}{dr}\right) + \frac{\mu}{r}\frac{dv}{dr}, \qquad \frac{\partial p}{\partial r} = \frac{\partial \mu}{\partial x}\frac{dv}{dr}, \qquad a\Delta T = v\frac{\partial T}{\partial x} \tag{1.2}$$

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We will assume that the tube wall is maintained at a temperature which depends linearly on x

$$T(x, r_0) = Ax \tag{1.3}$$

Here r_0 is the tube radius, and x is the longitudinal coordinate.

In the case of steady heat processes relationship (1.3) is identical to the assumption of constancy of the specific heat flux through the tube wall if ρ and c_p are constant over the cross section and length of the tube.

We introduce the dimensionless quantities

$$\begin{gathered} X = x/r_0,\ R = r/r_0,\ \theta = T/Ar_0,\ V = v/u_0 \\ P = pr_0/\mu_0 u_0,\ \mathrm{Pe} = u_0 r_0/a,\ \alpha = \beta A r_0 \end{gathered} \tag{1.4}$$

Here u_0 is the average flow velocity.

We assume that the temperature distribution is self-similar relative to coordinate X, i.e.,

$$\theta(R, X) = X + \theta(R) \tag{1.5}$$

In view of assumption (1.5) we obtain a dimensionless system of ordinary differential equations for the functions $\theta(R)$ and $V(R)$

$$\begin{gathered} \frac{d}{dR}\Delta V + \left[\alpha^2\left(\frac{d\theta^2}{dR} - 1\right) - \alpha \mathrm{Pe} V\right]\frac{dV}{dR} - 2\alpha\frac{d\theta}{dR}\frac{d^2V}{dR^2} = 0 \\ \Delta\theta = \mathrm{Pe} V \end{gathered} \tag{1.6}$$

The boundary conditions and conditions for constant flow rate have the usual form

$$V(1) = 0, \qquad \theta(1) = 0, \qquad \int_0^1 RV(R)\,dR = 1/2 \tag{1.7}$$

Relationships (1.6) and (1.7) are the mathematical formulation of the problem.

We formally seek the solution of Eqs. (1.6) in the form of a series of powers of the parameter α

$$V(R, \alpha) = \sum_{k=0}^{\infty} \alpha^k V_k(R), \qquad \theta(R, \alpha) = \sum_{k=0}^{\infty} \alpha^k \theta_k(R) \tag{1.8}$$

To determine the k-th approximation we obtain a system of linear differential equations

$$\begin{gathered} (d/dR)\,\Delta V_k = F_k(\theta_0, \theta_1, \ldots, \theta_{k-1}, V_0, V_1, \ldots, V_{k-1}), \qquad \Delta\theta_k = \mathrm{Pe} V_k \\ F_k = -\sum_{m=0}^{k-2}\sum_{n=0}^{k-2}\left(\frac{d\theta_m}{dR}\frac{d\theta_n}{dR}\frac{dV_l}{dR} - \frac{dV_l}{dR}\right) + \sum_{m=0}^{-1}\left(\mathrm{Pe} V_m \frac{dV_l}{dR} + 2\frac{d\theta_m}{dR}\frac{d^2V_l}{dR^2}\right) \end{gathered} \tag{1.9}$$

When the expansions (1.8) are taken into account conditions (1.7) take the form

$$\begin{gathered} V_k(1) = 0,\ \theta_k(1) = 0, \quad k = 0, 1, 2, \ldots \\ \int_0^1 RV_0(R)\,dR = 1/2, \qquad \int_0^1 RV_k(R) = 0, \quad k = 1, 2, \ldots \end{gathered} \tag{1.10}$$

Thus, to solve the problem we use the method of successive approximations. As the zero approximation we take the functions [1]

$$V_0(R) = 2(1 - R^2),\ \theta_0(R) = -1/8\,\mathrm{Pe}\,(3 - 4R^2 + R^4) \tag{1.11}$$

Using the method of variation of constants we can easily obtain a general solution of the system of Eqs. (1.9)

$$V_k(R) = \int_0^R \frac{1}{\xi}\int_0^\xi \eta \int_0^\eta F_k(y)\,dy + A_k R^2 + B_k$$
$$\theta_k(R) = \mathrm{Pe}\left(\int_0^R \frac{1}{\xi}\int_0^\xi \eta V_k(\eta)\,d\eta + C_k\right) \tag{1.12}$$

The arbitrary constants A_k, B_k, and C_k are determined from relationships (1.10)

$$A_k = 4\int_0^1 RX_k(R)\,dR - 2X_k(1)$$
$$B_k = -4\int_0^1 RX_k(R)\,dR + X_k(1) \tag{1.13}$$
$$C_k = -\int_0^1 \frac{1}{\xi}\int_0^\xi \eta V_k(\eta)\,d\eta$$

where $X_k(R)$ is a particular solution of the first equation in (1.9).

2. We assume that functions (V_0, θ_0), (V_1, θ_1), $(V_2, \theta_2), \ldots, (V_{k-1}, \theta_{k-1})$ have no singularities in the considered flow region. The analyticity of the k-th approximation then follows from Eq. (1.12). Since the zero pair of functions (1.11) is analytical in the region $(0 \le R \le 1)$, then approximations of any order and their derivatives will possess this property.

We turn now to the proof of uniform convergence of $V(R, \alpha)$ and $\theta(R, \alpha)$ and their derivatives in the range $0 \le \alpha \le \alpha^*$.

We introduce a positive number M_k, such that

$$\max |F_k(R)| \leqslant M_k \quad (0 \leqslant R \leqslant 1) \tag{2.1}$$

The introduction of M_k is always possible since we are dealing with functions which are continuous in a closed interval. From relationships (1.12) and (1.13) we obtain estimates for functions $\theta_k(R)$ and $V_k(R)$ and their derivatives to the first and second order inclusive

$$|V_k(R)| \leqslant 8/9\, M_k, \quad |\theta_k(R)| \leqslant 4/9\, M_k \cdot \mathrm{Pe}$$
$$|dV_k/dR| \leqslant 11/9\, M_k, \; |d\theta_k/dR(R)| \leqslant 4/9\, M_k \mathrm{Pe}, \; |d^2V_k/dR^2(R)| \leqslant 20/9\, M_k \tag{2.2}$$

Denoting by h the largest of the numbers (20/9, 4/9 Pe) we construct the series

$$W(\alpha) = \sum_{k=0}^{\infty} W_k \alpha^k, \quad W_k = hM_k \tag{2.3}$$

By α here we mean the modulus of this quantity.

If the convergence of function $W(\alpha)$ is proved, then the uniform convergence of expansions (1.8) and their derivatives in the region $0 \le R \le 1$ will follow from inequalities (2.2).

As still undetermined values of M_k we take the numbers

$$M_k = W_k/h = \sum_{m=0}^{k-2} \sum_{n=0, m+n+l=k-2}^{k-2} (W_m W_n W_l + W_l) + \sum_{m=0, m+l=k-1}^{k-1} (\mathrm{Pe} + 2) W_m W_l \tag{2.4}$$

If each term of series (2.3) apart from the zero term is replaced by its expression (2.4) we obtain the majorant equation

$$\varphi(W, \alpha) = \alpha^2 W^3 + \alpha(\mathrm{Pe} + 2) W^2 + (\alpha^2 - 1/h) W + W_0/h = 0 \tag{2.5}$$

where W_0 is the largest number exceeding the maxima of functions $V_0(R)$ and $\theta_0(R)$ and their derivatives.

Since the coefficients of $W(\alpha)$ are rational functions of α, then $W(\alpha)$ is an algebraic function and in the vicinity of a special point $(\alpha = 0)$ has at least one regular convergence branch. The nearest special point (radius of convergence) is determined after elimination of W from the system of equations

$$\varphi(W, \alpha^*) = 0, \; \partial\varphi / \partial W (W, \alpha^*) = 0 \tag{2.6}$$

In the vicinity of zero value of the parameter $0 \le \alpha \le \varepsilon$ we can replace Eq. (2.5) by the simpler equation

$$\alpha(\mathrm{Pe} + 2) W^2 - h^{-1}W + h^{-1}W_0 = 0 \tag{2.7}$$

In this case the radius of convergence is given by the root of the discriminant equation

$$\alpha^* = [4hW_0(\mathrm{Pe} + 2)]^{-1} \tag{2.8}$$

If, for instance, Pe = 1, then h = 20/9, W_0 = 4, and α^* = 0.01.

It is obvious that if more accurate estimates are made in inequalities (2.2) the value of α^* may be greater.

3. To determine the first approximation we have the system of equations

$$\frac{d}{dR}\Delta V_1 = \mathrm{Pe}\, V_0 \frac{dV_0}{dR} + 2\frac{d\theta_0}{dR}\frac{d^2V_0}{dR^2}, \qquad \Delta\theta_1 = \mathrm{Pe}\, V_1 \tag{3.1}$$

The solution of this system, which has no singularities at R = 0, is

$$\begin{aligned} V_1(R) &= {}^1/_{24}\,\mathrm{Pe}\,(2R^6 - 12R^4 + 13R^2 - 3) \\ \theta_1(R) &= {}^1/_{2304}\mathrm{Pe}^2\,(3R^8 - 32R^6 + 78R^4 - 72R^2 + 23) \end{aligned} \tag{3.2}$$

On the basis of two approximations we can put the axial flow velocity in the form

$$V(R, \alpha, \mathrm{Pe}) = 2(1 - R^2) + {}^1/_{24}\,\alpha\,\mathrm{Pe}\,(2R^6 - 12R^4 + 13R^2 - 3) \tag{3.3}$$

Analyzing Eq. (3.3) we can draw the following conclusions. When $\alpha > 0$ (the walls heat the liquid) the velocity on the tube axis decreases, and in the vicinity of the walls increases in comparison with the parabolic flow regime, so that a fuller velocity profile is obtained. When $\alpha < 0$ (the walls are colder than the liquid) the flow picture will be the opposite, and the velocity profile assumes the characteristic extended form. These effects are enhanced when the product αPe increases. These conclusions are in good agreement with the experimental data obtained for water and MS-20 oils [1].

The temperature gradient at the tube wall for all values of X is given by the relation

$$d\theta / dR = \mathrm{Pe} / 2 \tag{3.4}$$

since

$$\left.\frac{d\theta_k}{dR}\right|_{R=1} = \mathrm{Pe}\int_0^1 RV_k(R)\,dR = 0, \qquad k \geqslant 1$$

For the mean temperature over the cross section we obtain

$$\theta_s = 2\int_0^1 \theta(R)V(R)R\,dR = -\frac{11}{48}\mathrm{Pe} + \frac{\alpha\,\mathrm{Pe}^2}{128} \tag{3.5}$$

Thus, for the Nusselt number referred to the tube diameter we have the equation

$$\mathrm{Nu}^{-1} = \theta_s[-2d\theta / dR]^{-1} = 11/48 - \alpha\mathrm{Pe} / 128 \tag{3.6}$$

It follows from Eq. (3.6) that when $\alpha > 0$, the Nusselt number and, hence, the heat transfer coefficient increase in comparison with the limiting value Nu = 4.364. When $\alpha < 0$ the Nusselt number and the heat transfer coefficient will be reduced.

For the hydraulic resistance coefficient we have

$$\lambda = |\tau_{\max}| / {}^1/_2\,\rho U_0^2 = |8 / \mathrm{Re} + {}^5/_6\,\alpha\mathrm{Pr}|\, e^{-\alpha X}$$

Here τ_{max} is the friction stress on the tube walls and ρ is the density of the liquid, Re = $U_0 r_0 \rho/\mu_0$ is the Reynolds number, Pr = $\mu_0/\rho a$ is the Prandtl number.

When $\alpha < 0$ the resistance coefficient increases rapidly with increase in length X.

We note an interesting point. If the parameter $\alpha \ll 1$, then in system (1.6) the term $\alpha^2 dV/dR$ can be neglected in comparison with the rest. Then the required V, θ/Pe, and Nu will be functions of the single criterion αPe

$$V = f_1(R, \alpha \mathrm{Pe}), \; \theta = \mathrm{Pe} f_2(R, \alpha \mathrm{Pe}), \; \mathrm{Nu} = f_3(\alpha \mathrm{Pe}) \tag{3.7}$$

An experimental verification of the last equation in (3.7) is of interest.

A calculation of the subsequent approximations presents no difficulty, since the right hand sides of Eqs. (1.9) for any k are polynomials and, hence, for each approximation we can obtain an exact situation.

The question of the existence and uniqueness of the solution of the system of Eqs. (1.6) for $\alpha > \alpha^*$ remains open.

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